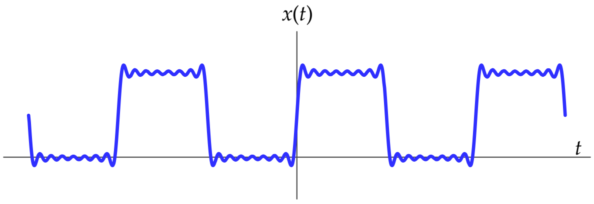
Fourier Series: The Concept

Prof. Mohammed Hawa
Electrical Engineering
The University of Jordan



Remember Maclaurin series of $x(t)$ around $t = 0$, which is the power series expansion

$$x(t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(0)}{n!}(t)^n = x(0) + \frac{x'(0)}{1!}t + \frac{x''(0)}{2!}t^2 + \frac{x'''(0)}{3!}t^3 + \cdots$$

For example, for the exponential function around $t = 0$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

Hence, in certain cases we can represent a particular signal using the sum (could be infinite sum) of other signals.

In **Fourier series**, **periodic signal** $x(t)$ can be expressed as the infinite sum of sinusoidal (cosine and/or sine) signals (called **harmonics**) plus an extra DC component (average value),

$$x(t)_{periodic} = \mathrm{DC} + \sum_{n=1}^{\infty} \text{sinusoids}(\omega_n = n\omega_0)$$

The **frequency** of the nth sinusoid **(nth harmonic)** is an integer multiple $n\omega_0$ of the fundamental frequency $\omega_0 = 2\pi/T_0$ of $x(t)$.

The **amplitude** and **phase** of the nth sinusoid (nth harmonic) are dependent on the actual periodic signal $x(t)_{periodic}$.

There are **three** mathematical equations to express this sum of sinusoids, each has its advantages and disadvantages.

Complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \ , \qquad \omega_0 = \frac{2\pi}{T_0}$$

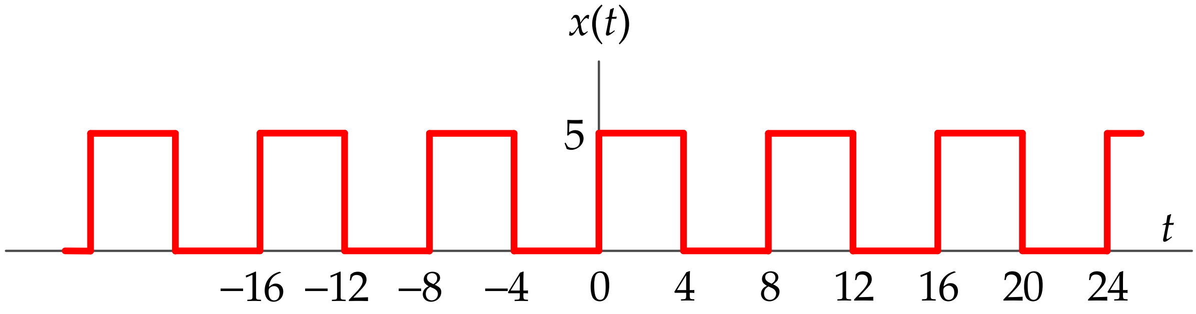
Trigonometric Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \ , \quad \omega_0 = \frac{2\pi}{T_0}$$

Compact Fourier series

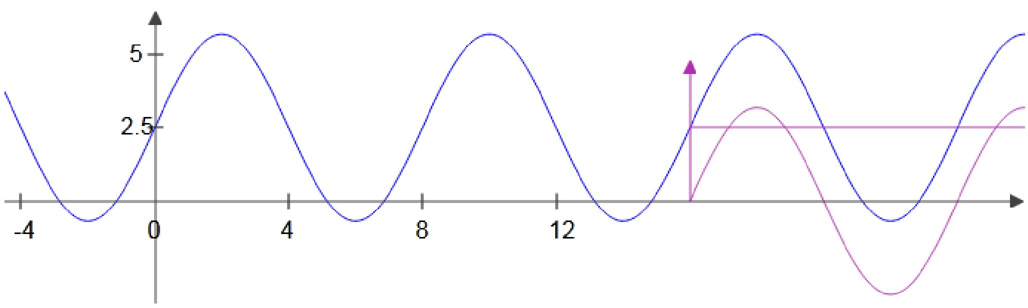
$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \ , \qquad \omega_0 = \frac{2\pi}{T_0}$$

Example: The following periodic signal $x(t)$ can be written as infinite sum of sinusoids

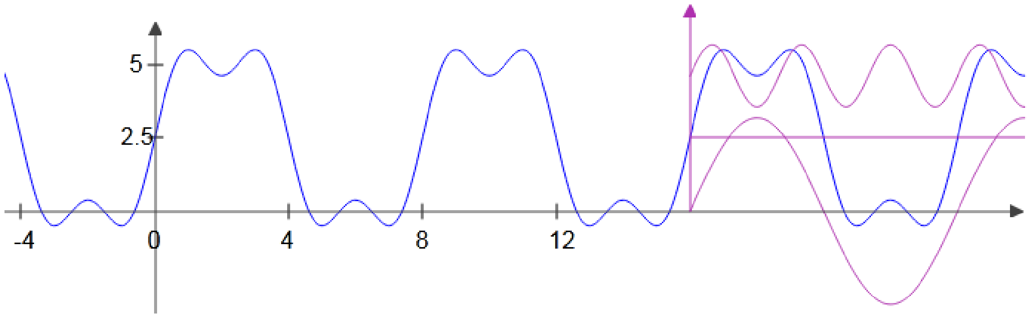


$$x(t) = \text{rep}_8\left\{5\ \text{rect}\left(\frac{t-2}{4}\right)\right\}$$

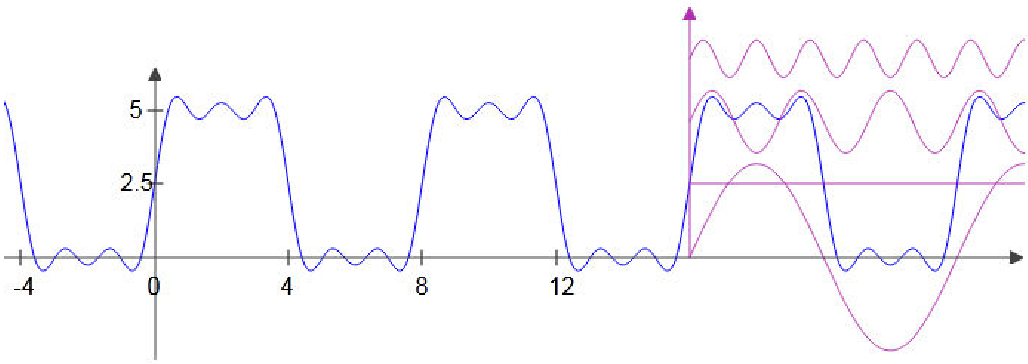
$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right)$$



$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{3\pi}\cos\left(\frac{6\pi t}{8} - \frac{\pi}{2}\right)$$



$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{3\pi}\cos\left(\frac{6\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{5\pi}\cos\left(\frac{10\pi t}{8} - \frac{\pi}{2}\right)$$



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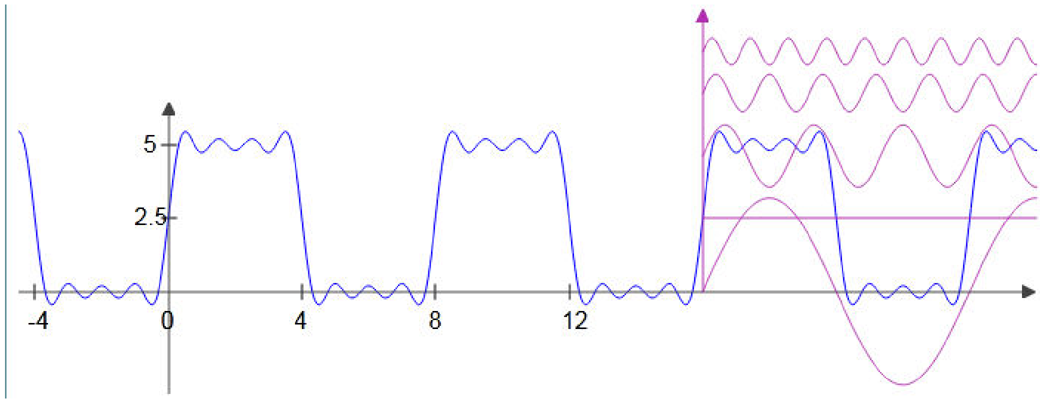
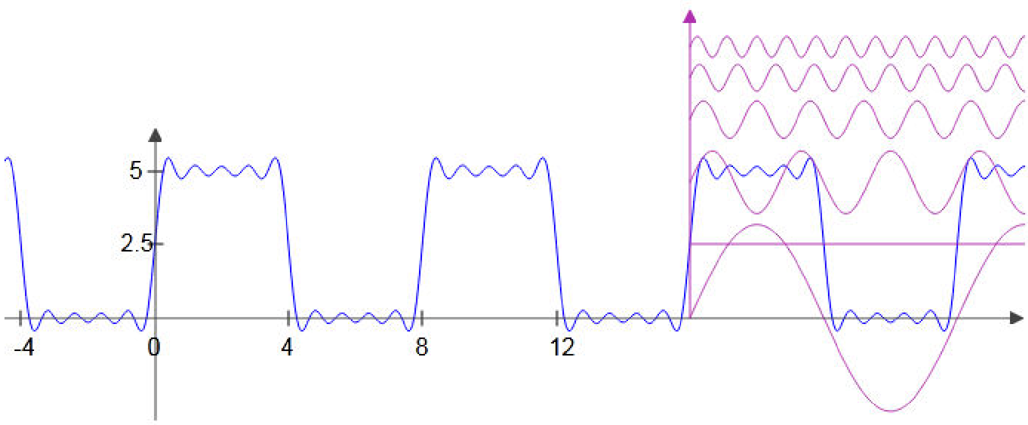


$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{3\pi}\cos\left(\frac{6\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{5\pi}\cos\left(\frac{10\pi t}{8} - \frac{\pi}{2}\right)$$
$$+0 + \frac{10}{7\pi}\cos\left(\frac{14\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{9\pi}\cos\left(\frac{18\pi t}{8} - \frac{\pi}{2}\right)$$



$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{3\pi}\cos\left(\frac{6\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{5\pi}\cos\left(\frac{10\pi t}{8} - \frac{\pi}{2}\right)$$

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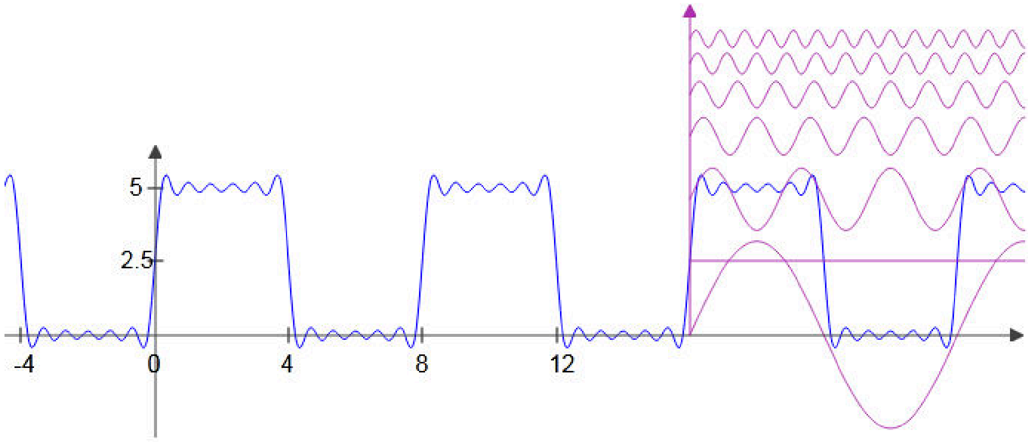
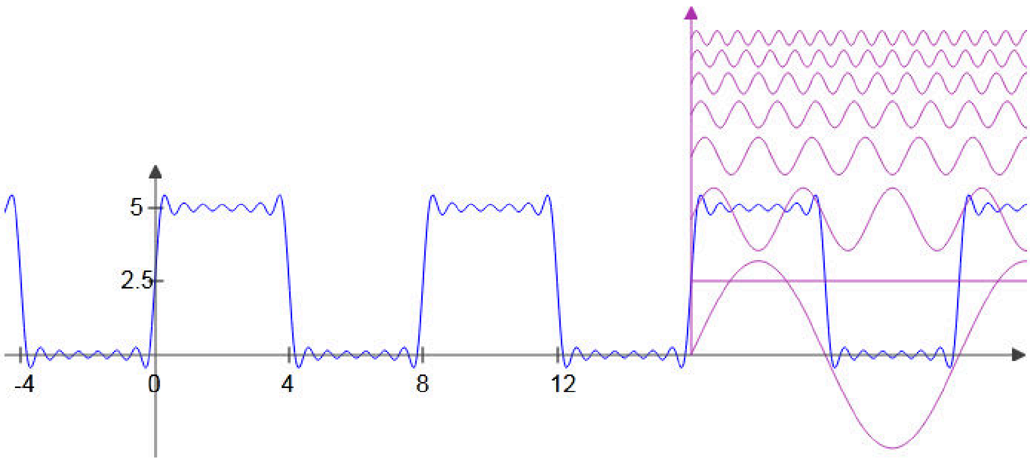
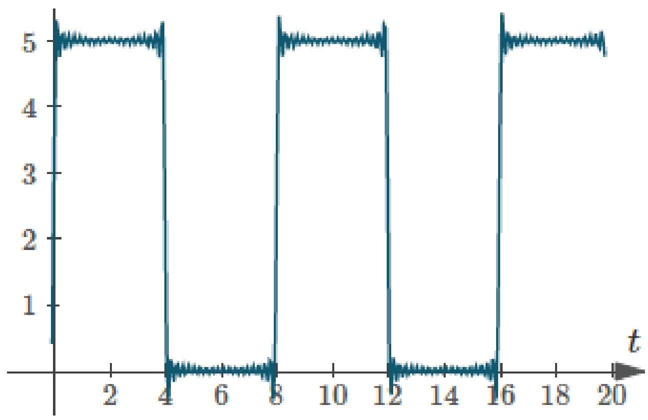


$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 2.5 + \frac{10}{\pi}\cos\left(\frac{2\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{3\pi}\cos\left(\frac{6\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{5\pi}\cos\left(\frac{10\pi t}{8} - \frac{\pi}{2}\right)$$

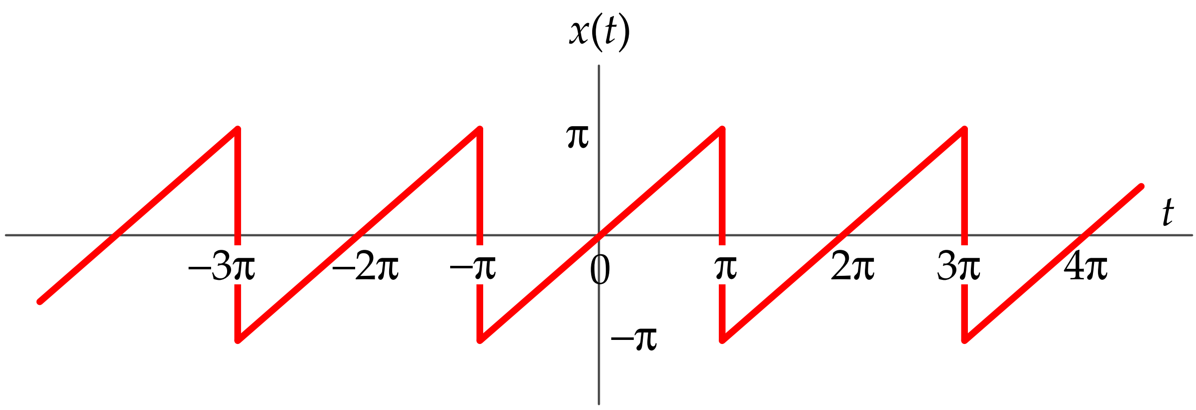
$$+0 + \frac{10}{7\pi}\cos\left(\frac{14\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{9\pi}\cos\left(\frac{18\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{11\pi}\cos\left(\frac{22\pi t}{8} - \frac{\pi}{2}\right) + 0 + \frac{10}{13\pi}\cos\left(\frac{26\pi t}{8} - \frac{\pi}{2}\right)$$



After adding 40 terms (20 zeros plus 20 cosines)

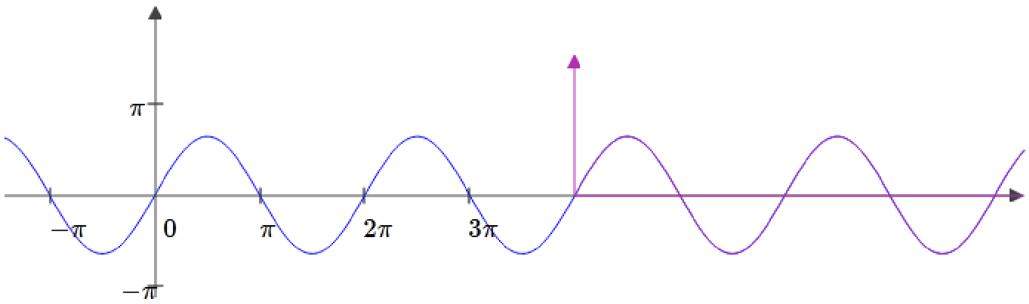


Example: The following periodic signal $x(t)$ can be written as infinite sum of sinusoids

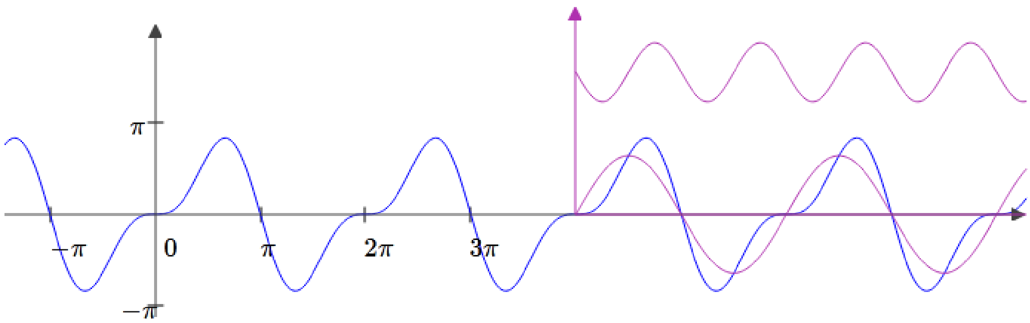


$$x(t) = \text{rep}_{2\pi}\left\{\pi \text{ saw}\left(\frac{t}{2\pi}\right)\right\}$$

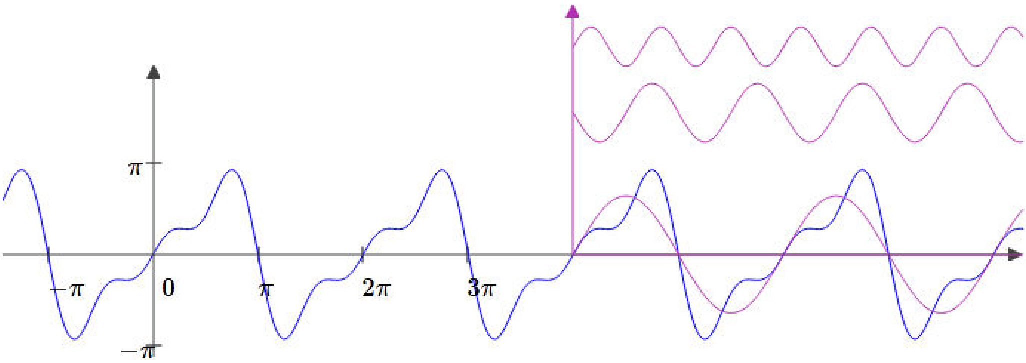
$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 0 + 2\cos\left(t - \frac{\pi}{2}\right)$$



$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 0 + 2\cos\left(t - \frac{\pi}{2}\right) + \cos\left(2t + \frac{\pi}{2}\right)$$



$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \approx 0 + 2\cos\left(t - \frac{\pi}{2}\right) + \cos\left(2t + \frac{\pi}{2}\right) + \frac{2}{3}\cos\left(3t - \frac{\pi}{2}\right)$$



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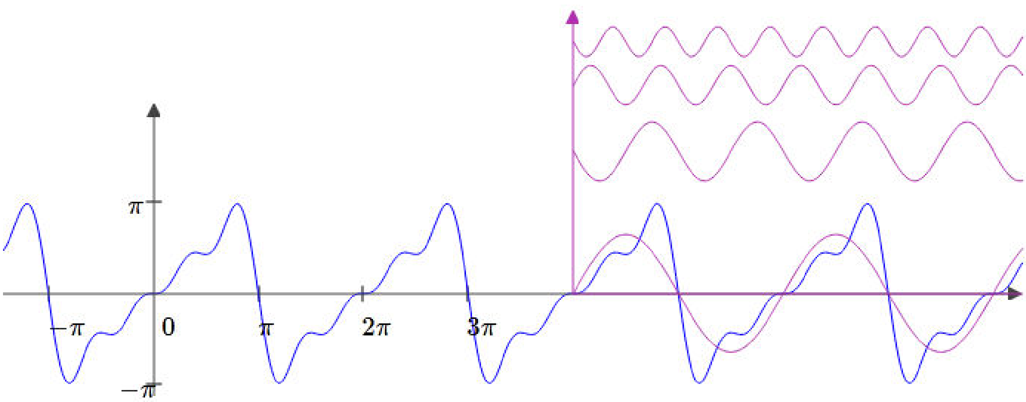

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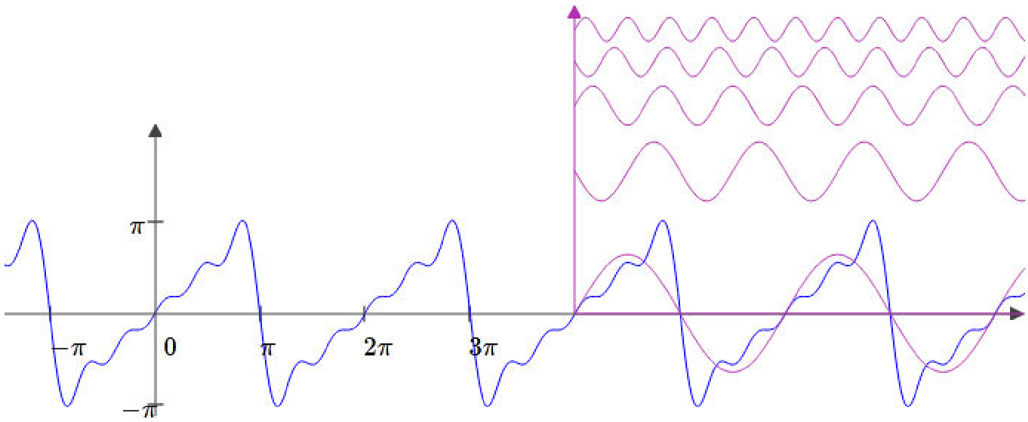

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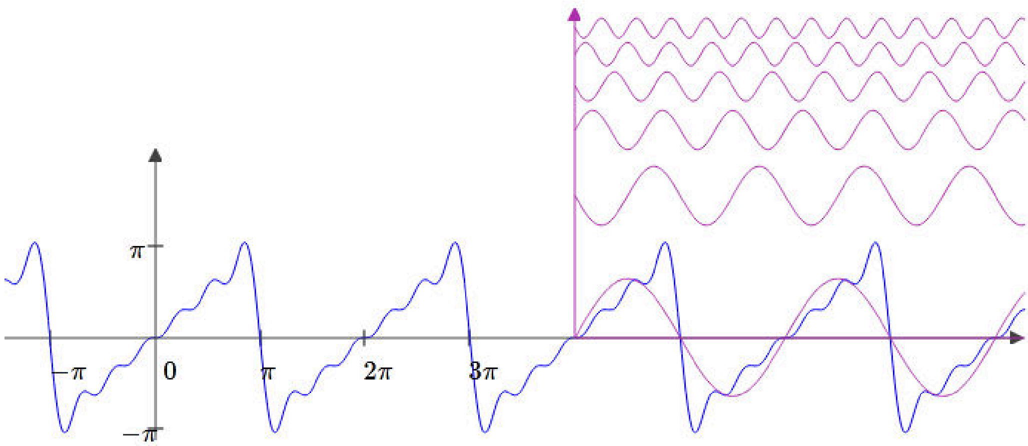
$$+ \frac{1}{3}\cos\left(6t + \frac{\pi}{2}\right)$$



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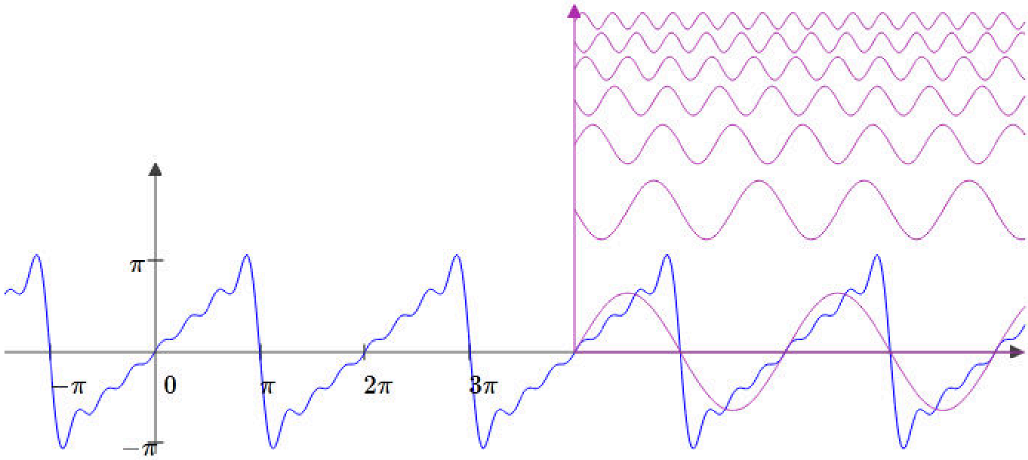
$$+\frac{1}{3}\cos\left(6t + \frac{\pi}{2}\right) + \frac{2}{7}\cos\left(7t - \frac{\pi}{2}\right)$$



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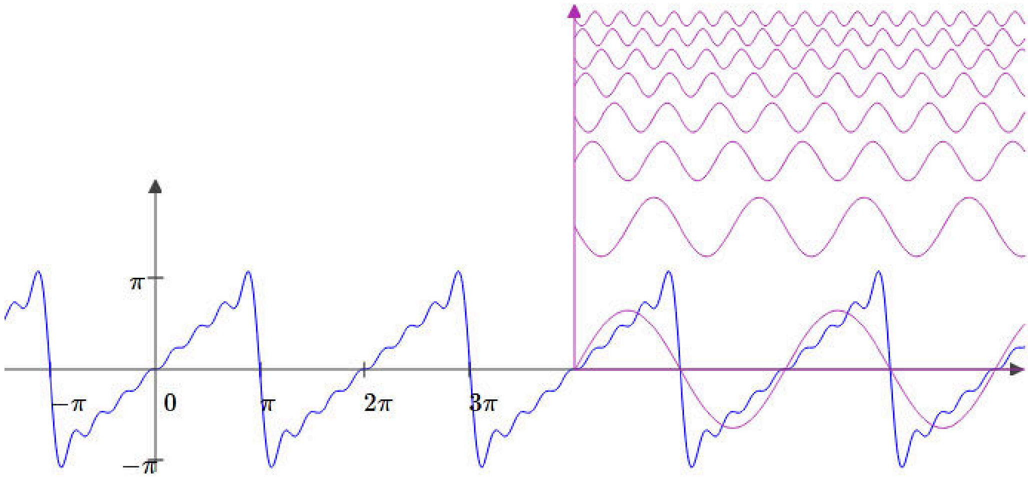
$$+\frac{1}{3}\cos\left(6t + \frac{\pi}{2}\right) + \frac{2}{7}\cos\left(7t - \frac{\pi}{2}\right) + \frac{1}{4}\cos\left(8t + \frac{\pi}{2}\right)$$



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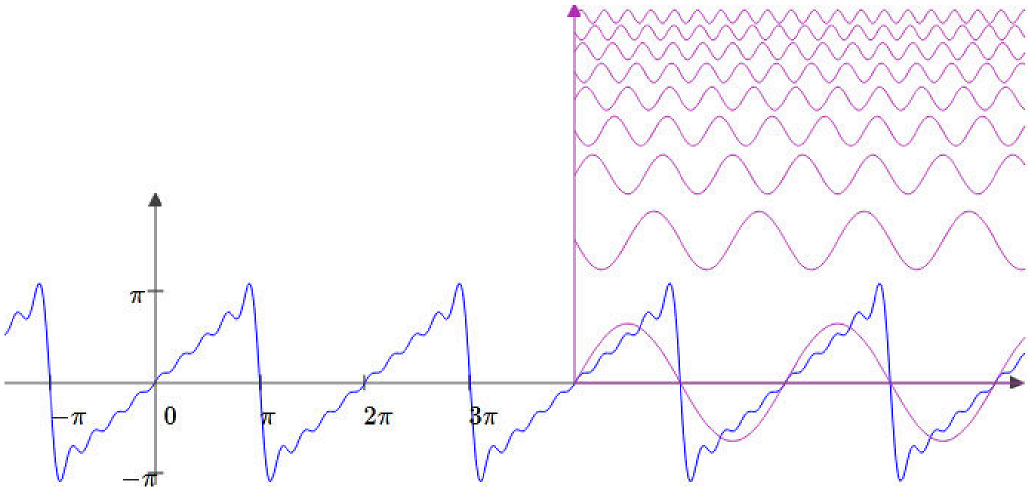
$$+ \frac{1}{3}\cos\left(6t + \frac{\pi}{2}\right) + \frac{2}{7}\cos\left(7t - \frac{\pi}{2}\right) + \frac{1}{4}\cos\left(8t + \frac{\pi}{2}\right) + \frac{2}{9}\cos\left(9t - \frac{\pi}{2}\right)$$



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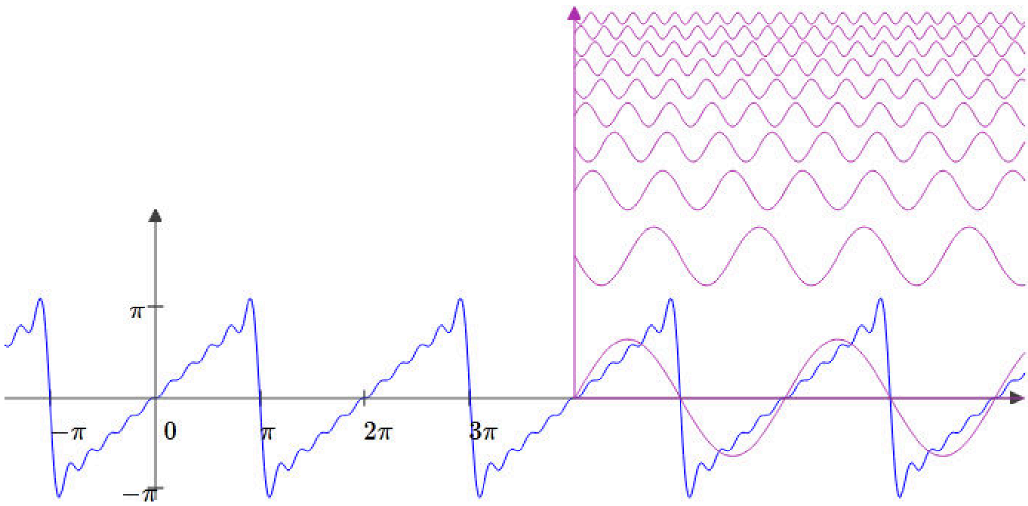
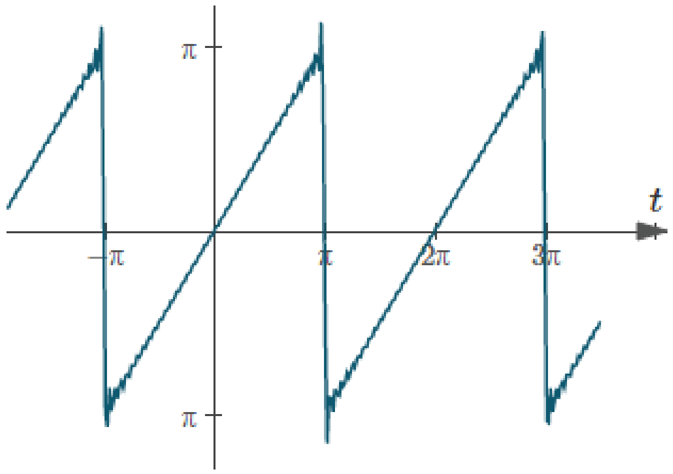
$$+ \frac{1}{3}\cos\left(6t + \frac{\pi}{2}\right) + \frac{2}{7}\cos\left(7t - \frac{\pi}{2}\right) + \frac{1}{4}\cos\left(8t + \frac{\pi}{2}\right) + \frac{2}{9}\cos\left(9t - \frac{\pi}{2}\right) + \frac{1}{5}\cos\left(10t + \frac{\pi}{2}\right)$$



After adding 40 terms (cosines)



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Compact Fourier series

$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \varphi_n) \,, \qquad \omega_0 = \frac{2\pi}{T_0}$$